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# Contingency and collapse in credit default swap networks

by Roger J. Bowden[1]

## Abstract

Credit insurance via credit default swaps should theoretically improve market efficiency and lower the cost of capital by allowing credit risks to be shared by a wider market. In practice, apparent risk sharing led to a faster collapse: the subprime crisis and consequential credit crisis. By modelling the counterparty system as a two dimensional network, the matrix representation can be used to identify points of weakness for counterparty survival and hence the survival of principals who have bought credit protection. Useful analogies exist with input output economics and econometrics. Counterparty portfolios that are collinear or concentrated are an implicit exposure to apparently unrelated parties, predisposing the system to contagious collapse. Policy implications for credit rating agencies and regulators are explored.

Key words: Contagion, counterparty survival, credit default swaps, networks, structural and reduced forms.

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[1] Kiwicap Research Ltd Wellington, New Zealand, and Institute of Finance, Faculty of Mathematics and Economics, Ulm University, Germany. The author is grateful to Peter Posch for some useful comments on a working draft of the paper.

# I Introduction

The most startling thing about the subprime crisis was the speed of collapse, effectively in the course of a single month, June 2007. A second surprising thing (of many) was that so few otherwise very competent financial regulators thought there was any danger at the time[2]. On the face of it, the structures that evolved prior to the collapse collectively constituted an effective way to spread risk and therefore to lower the cost of debt capital for a spectrum of purposes, ranging from home mortgages and consumer finance to commercial lending. Even after the collapse, much the same instruments continue to exist and are actively traded. Banks continue to hold credit default swaps (CDS) in their trading books and as enhancements or synthetic bonds in their loans book; and the same hedge fund managers who rushed to dump CDS have now quietly commenced rebuilding their portfolios. All this suggests that the problem was not so much the instruments, or even the immediate structures or investment vehicles built upon them, as an inherent property of the system that provided the scaffolding for their use. Why, exactly, did the framework collapse so quickly and comprehensively, and what safeguards need to be in place to forestall a repeat episode?

One answer to the latter question is an ever more detailed and prescriptive regulatory rulebook[3], or to require so much capital and/or liquidity to be allocated that the benefits of risk spreading are entirely lost. But all this may amount to so much deadweight cost unless it manages to locate and address the central problem, namely the inherent tendency of a debt pyramid built out of CDS to collapse. In this paper the system of physical debt and CDS is modelled as a dynamic network, the matrix representation of which enables a more detailed identification of the inherent problems and what can be done to fix them or to limit their effect. The bulk of the research that has been done on CDS and related structures has been concerned with the financial technology of no-arbitrage pricing in a locally complete context. The difficulty with this is that it cannot handle the wider aspects of counterparty risk: how it arises, how it spreads, and the contagion thus created. A network approach can be

[2] See e.g. Lewis (2010), Ishikawa (2009) and numerous web sources on famous last words from regulators; even Youtube weighs in. In fairness, it has to be said that regulators have a vested interest in not causing a panic with public announcements or responses to interviewers; what is of more relevance is the failure to act. Some regulators were actively encouraging banks to enter into the credit protection seller business, seen as an effective way to lever up tier 1 bank capital.

[3] Even before the collapse, the UK Financial Services Authority rulebook for financial institutions ran to 8,500 pages. The general expectation for Basel III and associated directives is for yet more ‘granularity’, in the official jargon. The proposed European Security Markets Authority (ESMA) is just one of a number of national and supranational new regulatory bodies.

informative for both pricing and regulation because it does identify what can go wrong, and suggest controls or safeguards that could be put into place to limit the fallouts.

Some instructive analogies result with other areas of economic or econometric research or practice. A CDS network is analogous to an input-output system, and the same terminology usable for both. Thus there is the idea of a fully simultaneous system decomposable in some cases into a recursive one where one class of agents buys credit protection and another class of agents sells it. Their portfolio behaviour in either case can display the same weakness as collinearity in econometrics, with similar recommendations as to how avoid it in terms of both structural and reduced form models (to continue the analogy). In the current context, this amounts to an imperative either for diversification in CDS exposure or for sharing to limit the overall effect of defaults in the underlying physical debt. The pre-crisis CDS portfolio of AIG was the best known violation of this principle, but according to the present analysis, the fault lay not only with AIG but also with its counterparties: portfolio diversification or exposure dilution has to apply to agents on both sides, buying and selling. Lewis (2010) is a very readable account of the failures referred to. In turn, diversification and related failures have to be recognised and priced, even if only implicitly via the ratings assigned by credit rating agencies. Likewise, a role for the regulatory agencies is to lay down rules that cover recognition and rectification of potential diversification problems and excessive exposure shares.

The CDS network as modelled in the present paper has a dual default liability, structured around the impact of deteriorating quality of the underlying physical debt upon the survival value of CDS counterparties. Counterparty survival depends upon net income from the swap, encompassing interest coupon, swap premium and the contingent payout liability or receipt from the swap. If this falls below capital reserves, then counterparty default occurs or is more likely to occur. In a poorly structured system, downgrade or default in an underlying physical debt issue can easily lead to survival stresses, even to parties who do not hold that particular debt on their books. Of specific interest is what happens when the system is subject to stress. Both structural and reduced forms can be used to establish conditions under which such stress can be terminal for the system as a whole.

The scheme of the rest of the paper is as follows. Section II sets up the model, which incorporates premium and interest income and contingent flows, collectively in the form of a network matrix of exposures. Counterparty survival determinants are modelled in section III, with a classification of system structures and predispositions to structural weakness. Section IV turns to sensitivity analysis, the extent to which physical default in one asset can lead to systematic default in other sectors arising from common exposures to swap counterparties. The role of portfolio concentration in system dynamics is explored by setting up a formal metric for exposure shares. Section V contains some concluding remarks, and explores policy implications.

## II The model and the counterparty matrix

A structure that encompasses CDS written on a set of primary risks (in this case, security downgrades or defaults) has in principle three dimensions: the primary risk (default on the physical debt), the agent seeking to lay off that risk (credit protection buyer), and the agent taking on the risk in exchange for the swap premium (credit protection seller). Thus party $i$ contracts with party $k$ to lay off the risk on asset $j$. However, the need for tensor notation can be avoided by recasting the system in two dimensions, extending the dimensionality if necessary; an agent buying CDS risk protection on multiple primary risks is treated as a collection of individual agents, one for each risk.

Thus in the model that follows, one agent is formally associated with each primary risk, and its potential on-selling via credit default swaps. Agent $j$ might be holding physical bond $j$ as an asset and wish to buy credit risk protection through the swaps market, perhaps with a multiplicity of counterparties, each selling a portion of the credit risk protection. However, the model also allows agent $j$ to buy credit risk protection against asset $j$ even there is only partial or even zero exposure to the costs of default on the associated physical: ‘ uncovered risk protection’ or naked CDS, in market terminology. On the other hand, an individual agent $i$ can also sell credit risk protection against any number of primary risks $\{j\}$ associated with other parties. This would correspond to a fund of CDS as used to construct synthetic bonds.

Default or downgrade status in the underlying physical debt is denoted by the binary variable $d_j$ which has boundary values unity if default occurs, zero otherwise. Intermediate values indicate a downgrade, on some numerical scale completing the usual letter based metric (e.g. A3 to Baa1 in Moody’s). Thus $d_j$ is a continuous variable taking values between 0 and 1, referring to the (inverse) changes in credit status over the period.

Credit default swaps can be written by party $j$ to an upper limit, or ‘nominated cost’, of $C_j$ if complete default of physical debt $j$ occurs. Under the downgrade interpretation the nominated cost will be $Cd_j$ . The term $d_j$ is often called the ‘loss given default’ (LGD), and the term $C_j$ the exposure at default (EAD).

The actual cost incidence for agent *j* of default in its allocated primary risk *j* is given by a number $\beta_j \in [0,1]$, such that agent *j* bears a direct physical cost $\beta_j C_j d_j$ if its allocated primary risk downgrades or defaults. Agent *j* can offset this by buying the same amount of credit risk protection, or even make contingent money by buying credit risk protection on the entire nominated amount $C_j$.

With the above conventions, the implied network can be represented as a nonnegative square matrix $\{A = ((\alpha_{ij})); 0 \leq \alpha_{ij} \leq 1),\ j = 1,2,...n\}$ where $\alpha_{ij}$ denotes agent *i*'s share in agent *j*'s primary risk under a CDS concluded between the two. Under this arrangement, if primary risk *j* defaults completely, agent *i* will pay $\alpha_{ij} C_j$ to agent *j*, or more generally, $\alpha_{ij} C_j\, d_j$ . If $\alpha_{jj} > 0$ this is taken to mean that agent *j* both pays to, and receives from, itself under the swap, an artifice to normalise the swap matrix. The columns of *A* add to unity: $\sum_{i=1}^{n} \alpha_{ij} = 1$, or collectively $A'\mathbf{1} = \mathbf{1}$ where $\mathbf{1}$ is the unit or summation vector. The actual cost to party *j* of default in its assigned primary risk *j* is captured by the term $\beta_j C_j$. As earlier indicated, uncovered or 'naked' swaps will arise where $\beta_j < 1$.

Expression (1) is an illustration. The chosen agent, party # 2, is sponsor or owner for physical debt #2. Assume it has an actual monetary exposure of \$50m ($\beta_{22}\times$100m) if its physical does default, where the \$100m is the nominated amount of the debt, e.g. the total on issue. Party 2 purchases credit protection against the nominated value of that debt to the proportions of 0% from party 1, 50% from party 2 and 25% from party 3. These are entered in column 2. In the event of complete default ($d_2 = 1$) on a nominated total of $C_2$ = \$100m, party 2 would receive \$50m from party 3 and #25m from party 4. The remaining 25%, i.e. \$25m is treated as a receipt from party 2 to itself, contingent on default of its associated physical. Party 2 can also engage in selling credit protection and this is represented in row 2. It has insured 30% of party 1, 20% of party 3 and 50% of party 4. Party 2 has also undertaken to meet 25% of its own contingent physical default, which nets out to zero with its role as protection buyer. In summary, party 2 will collect a net (0.5+0.25)×100m -50m = \$25m if its associated debt defaults. On the other hand, it will have potentially to pay out $0.3C_1 + 0.2C_3 + 0.5C_4$ if the debt sponsored by parties 1,3,4 all default.

$$
A = \begin{array}{c} \leftarrow \text{protection sellers} \rightarrow \end{array}
\overset{\leftarrow \text{ Debt } \otimes \text{ protection buyers } \rightarrow}{
\begin{bmatrix} * & 0 & * & * \\ 0.3 & 0.25 & 0.2 & 0.5 \\ * & 0.5 & * & * \\ * & 0.25 & * & * \end{bmatrix}}
\qquad \beta_{22} = 0.5 \quad .
$$

$$
\begin{array}{cccc} \overline{1.0} & \overline{1.0} & \overline{1.0} & \overline{1.0} \end{array}
$$

Figure 1: Structure of the swap counterparty matrix

Credit default swaps have a nominal maturity of one period. At the end of that period, payment is due on any primary risk that has defaulted or downgraded over the period. However, payments under a swap are conditioned by whether or not the credit protection seller survives, i.e. is capable of meeting all the end of period payments. The burden of its commitments in this respect may well drive it bankrupt. In turn, if it is unable to meet the payment due to be made to the party who has bought protection, then the latter party may also not survive following a default on its physical debt holding. There is no intra period recontracting in this model, so if a party loses its risk protection through counterparty default it is then wholly exposed to default in its physical.

Survival of party *j* is denoted by a binary variable $s_j$, which has value unity if party *j* survives and zero if ('demise'). The possibility and meaning of intermediate values is explored at a later point, together with survival conditions.

With the above conventions, the receipts and payments to each party under the CDS network are as follows:

Contingent receipts:

(1a) $$[C_j d_j \alpha_{jj}] + C_j d_j \sum_{i \neq j} \alpha_{ij} s_i ; \quad j = 1,2,...n.$$

Contingent payments:

(1b) $$[C_j d_j \alpha_{jj}] + \sum_{i \neq j} \alpha_{ji} C_i d_i + \beta_j C_j d_j ; \quad j = 1,2,...n.$$

The square bracketed terms in expressions (1a,b) are the notional self-receipts and payments where $\sum_{i \neq j} \alpha_{ij} < 1$. The second term in (1a) refers to payments made to agent *j* by its counterparty credit protection sellers $\{i \neq j\}$ in the event of downgrade or default on its associated physical. The second term in 1(b) captures the contingent payments necessary where agent $j$ has sold credit protection on other contingent risks associated with agents $\{i \neq j\}$. In addition, agent $j$'s physical cost arising from a downgrade on its primary risk is given by the term $\beta_j C_j d_j$.

Referring to expression (1b) for payments, one can alternatively write:

$$\sum_{i \neq j} \alpha_{ji} C_i d_i + \beta_j C_j d_j = (\beta_j - \alpha_{jj}) C_j d_j + \sum_{i=1}^{n} \alpha_{ji} C_i d_i \,.$$

If $\beta_j = 1$, then agent $j$'s residual exposure to the physical debt is $(1-\alpha_{jj})$. If $\beta_j = \alpha_{jj}$ the effect is as if agent j has to make contingent payments to every other agent, including itself. If $\beta_j = 0$ then agent j has no physical contingent cost and is engaged only in buying credit protection from other parties; e.g. a hedge fund betting on defaults and willing to pay swap premiums in the meantime.

Swap premiums, denoted $\pi_j$, are paid up front and are certain, i.e. independent of contingent survival. To maintain temporal comparability with end of period contingent payouts on the swaps, one could imagine the premiums are sequestered in a risk free fiduciary account, and accrue interest, incorporated in an equivalent end of period premium. The swap premiums will in general vary with the default probabilities and losses or expected recovery on the underlying primary risk, i.e. the physical bonds, but for present purposes will be taken as predetermined; further remarks on this aspect are in section V.

Holders of primary risk $j$ will receive interest at rate $r_j$ on their physical bonds, treated as contingent, so that no interest is received if bond $j$ defaults. It will be convenient to suppose that this is a yield on the amount at risk, taken as $C_j$. The amount received will also depend upon party $j$'s degree of exposure to the physical, measured as $\beta_j$.

The income received from interest coupons and swap premiums is therefore given by

(2a) $\quad \beta_j r_j C_j (1-d_j) + \sum_{i \neq j} \pi_i \alpha_{ji}\,; \quad j = 1,2,...n,$

while income paid out is

(2b) $\quad \pi_j \sum_{i \neq j} \alpha_{ij} = \pi_j (1-\alpha_{jj})\,; \quad j = 1,2,...n.$

Taking the difference (2a)-(2b) gives the net premium income for agent j, which can be written collectively over $j = 1,2,...n$ as $(A-I)\boldsymbol{\pi}$. The sum over agents is $\boldsymbol{1}'(A-I)\boldsymbol{\pi} = 0$; swaps are pure exchange, where $\boldsymbol{1}' = (1,1,...1)$ is the unit or summation vector.

Combining expressions (1) and (2), agents' net end of period receipts and payments ('net income' for brevity) from all sources are given by

(3a)

$$y_j = \beta_j r_j C_j(1-d_j) + C_j d_j \sum_{i\neq j}\alpha_{ij}s_i - (\beta_j - \alpha_{jj})C_j d_j - \sum_{i=1}^{n}\alpha_{ji}C_i d_i + \sum_{i\neq j}\alpha_{ji}\pi_i - \pi_j; \quad j = 1,2,...n.$$

This can be put into matrix-vector terms as:

(3b) $\boldsymbol{y} = \hat{B}\hat{C}(I-\hat{D})\boldsymbol{r} + \hat{C}\hat{D}(A-\hat{A})'\boldsymbol{s} - (A-\hat{A}+\hat{B})\hat{C}\boldsymbol{d} + (A-I)\boldsymbol{\pi}$,

where vector definitions and dimensions are implicit on the basis of system (3a).

The convention throughout[4] will be that a caret over the top of a symbol indicates a diagonal matrix, either assembled from a vector; e.g. $\hat{D} = diag(d_1, d_2,...d_n)$ ; $\hat{B} = diag(\beta_1, \beta_2,...\beta_n)$ ; $\hat{R} = diag(r_1, r_2,...r_n)$ ; or from the diagonal elements of a matrix, e.g. $\hat{A} = diag(\alpha_{11}, \alpha_{22},...\alpha_{nn})$.

In expression (3b), the term $\hat{C}\hat{D}(A-\hat{A})'\boldsymbol{s}$ incorporates receipts from buying risk protection, due from swap counterparties. The term $(A-\hat{A}+\hat{B})\hat{C}\boldsymbol{d}$ represents the contingent payment due to counterparties to whom credit protection has been sold.

Swap premiums are assumed to be set at the start of the period. However, one would expect[5] this to be a self financed system ( $\boldsymbol{y} \geq \boldsymbol{0}$ ), given the best state of the world as the end of period outcome, namely no defaults ($\boldsymbol{d} = \boldsymbol{0}$) and all parties survive ($\boldsymbol{s} = \boldsymbol{1}$). Substituting in (3b), this implies

$$\hat{B}\hat{C}\hat{R}\boldsymbol{1} \geq (I-A)\boldsymbol{\pi}.$$

The interpretation is that net swap payments have to be financed out of interest income on the underlying physical debt. In what follows, it is assumed that at least some of the elements $\beta_j$ of $\hat{B}$ are non zero, i.e. there is some exposure in the system to the underlying physical debt.

[4] There is a degree of freedom in using diagonal matrices in conjunction with vectors; thus in equation (3b), $\hat{D}\boldsymbol{r}$ could be written as $\hat{R}\boldsymbol{d}$ or as $\hat{D}\hat{R}\boldsymbol{1} = \hat{R}\hat{D}\boldsymbol{1}$, where **1** is the unit vector.

[5] In market jargon this state of affairs called a negative basis spread. In more speculative situations a positive basis spread can exists for some or all securities where the interest earnings do not cover the premiums being paid on the same asset. See Rajan et al (2007).

## III Counterparty survival

If the net payout or receipt $y_j$ falls below a certain threshold level relative to initial capital $k_j$, then swap counterparty $j$ defaults on its payment obligations. This can be modelled in terms of the conditional probability of default, given the vector $\boldsymbol{d}$ of primary (physical bond) defaults. For the purposes of the present section, capital reserves **k** are taken as given.

*3.1 Survival specifications*

Alternative specifications for the survival variable $s_j$ can be nested within an underlying random survival threshold centred on capital reserves. It is effectively a probit model with just one independent variable, namely net income, though more could be added in as in the literature on probits and logits for debt default (e.g. Posch (2007), Loeffler and Posch (2007)). Thus if $\varepsilon_j$ denotes a zero mean random variable, representing random elements assumed independent of physical debt default status, define a random variable

(4a) $$\begin{aligned}\tilde{s}_j &= 1 \Leftrightarrow y_j < -k_j + \varepsilon_j \\ &= 0, \quad \textit{otherwise.}\end{aligned}$$

The conditional expectation is given by

(4b) $$E[s_j \mid y_j] = u_j(y_j + k_j; \sigma_j),$$

where $\sigma_j$ is a variance parameter. The function $u$ is illustrated in Figure 2a with the survival threshold $k$ set as -1.0. As $\sigma \to 0$ the expectation becomes the discrete step or Heaviside function depicted in figure 2b.

Alternative specifications for actual survival $s_j$ can be cast in terms of the latent or generating variable $\tilde{s}_j$ :

(5a) $$s_j = \tilde{s}_j;$$

(5b) $$s_j = E[\tilde{s}_j \mid y_j];$$

(5c) $$s_j = u_j(y_j + k_j; \sigma_j).$$

Formulations (5b) and (5c) are formally identical, but in (5c) $s_j$ is deterministic in character, and the function $u_j$ is reinterpreted in the manner of Zadeh (1965,1996), as an imprecise or fuzzy switch, as distinct from the sharp of binary switch of figure (2b), the degree of impression being measured by the parameter $\sigma_j$ . This formulation is a possible way to capture the degree of default, referring to the possibility of partial payments being made.

The development that follows can refer to either (5b) or (5c), with (5a) as a limiting case where $\sigma_j \to 0$. The variables $s_j; j = 1,2,...n$ will generally be referred to as expected survival as in (5b), with the understanding that alternative interpretations are possible. Note that this is expected survival conditional upon net income, which in turn is conditional upon physical debt defaults ***d*** in the system. Expressions (5) together with (3b) form a simultaneous system of equations determining counterparty survival. The system is nonlinear and subject to rapid regime shifts because of the threshold nature of the relationship between survival and net income.

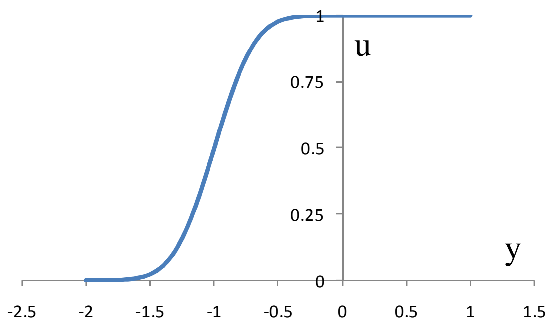


Figure 2a: Fuzzy survival

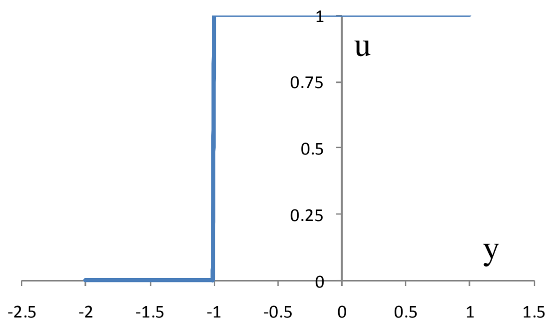


Figure 2b: Binary survival

*3.2 Survival and network structure*

Network interdependence arises as the result of causal links between physical bond defaults and survival of swap counterparties otherwise needed to make good those defaults. In a fully interdependent system, any specific agent could variously: (i) hold the physical, and be subject to default risk; (ii) suffer a failure in that protection because a swap counterparty does not survive the call; or (iii) write CDS protection on other physicals, and therefore be subject to calls on other physical defaults.

A fully interdependent system of the above kind corresponds to an irreducible network matrix *A*. On the other hand, a reducible network is one where the given network matrix *A* is similar via a renumbering or row and columns[6] to either a block triangular matrix or further to a block diagonal matrix (a fully decomposable system).

Thus consider the system

(6a) $$A = \begin{bmatrix} \hat{A}_{11} & O \\ A_{12} & A_{22} \end{bmatrix}; \quad A - \hat{A} = \begin{bmatrix} O & O \\ A_{21} & O \end{bmatrix},$$

[6] That is, there exists a permutation matrix *P* such that *PAP'* is block triangular.

where $\hat{A}_{11} = diag(\alpha_{11}, \alpha_{22} ... \alpha_{n_1})$, $A_{12}$ is $n_2 \times n_1$. This corresponds to a network where agents of type 1 ( $j=1,2,...n_1$) hold physical debt and buy credit protection from agents of type 2. The latter may also hold debt and may buy credit protection from other type 2 agents. Type 2 agents might correspond to banks making a market in CDS through their trading book, as well as holding the physical debt in their loan book.

An important special case of the general reducible form (6a) is

(6b) $A = \begin{bmatrix} \hat{A}_{11} & O \\ A_{12} & I \end{bmatrix}; \quad B_{22} = O$,

where will be recalled that the diagonal matrix $\hat{B}$ contains the physical cost exposures, with $\hat{B}_{22}$ containing those of the type 2 agents. The identity matrix appearing in (6b) is $n_2 \times n_2$ . In what follows, the identity matrix will be written with the dimension understood to conform to context (thus in (6b), just $I$ rather than $I_{11}$); similarly for the zero matrix, $O$.

According to specification (6b), type 2 agents are solely in the business of providing credit protection to type 1 agents; they are pure insurers operating via selling credit risk protection on CDS. A representative element $\alpha_{ij}$ of $A$ gives the proportion of contingent cost $C_j$ provided by type 2 agent $i$ to type 1 agent $j$.

Network survival risks arise even with recursive systems such as model (6b). Using expression (3b), the effect on net income is given for this recursive model by

(7a) $\boldsymbol{y}_1 = (\hat{B}\hat{C})_{11}\boldsymbol{r}_1 - (\hat{B}\hat{C}(I+\hat{R}))_{11}\boldsymbol{d}_1 + (\hat{C}\hat{D})_{11}A_{21}'\boldsymbol{s}_2 - (I-\hat{A})_{11}\boldsymbol{\pi}_1$

(7b) $\boldsymbol{y}_2 = -A_{21}\hat{C}_{11}\boldsymbol{d}_1 + A_{21}\boldsymbol{\pi}_1$.

The recursive nature of the specification is evident from expressions (7a,b). According to (7b), a downgrade in physical debt #1, say, would cause default-induced loss for the type 2 credit sellers, according to the exposures comprising the first column of $A_{21}$ . If these losses exceed their respective reserve buffers, then the credit sellers default on their swap obligations. In turn, this impacts on the survival prospects of agent #1, who is now unprotected against the given physical downgrade or default.

Indeed, if the default costs arising from just the one physical default trigger uniform demise in type 2 agents then $\boldsymbol{s}_2 = \boldsymbol{0}$ in the first equation (7a). All type 1 credit protection buyers lose their compensating payout, possibly triggering their own demise. Death is contagious.

## IV Sensitivity analysis

The system defined by (3b) and (5) is nonlinear in net income and survival. It is also simultaneous: net income $\boldsymbol{y}$ depends upon survival $\boldsymbol{s}$ in (3b), but survival depends upon income, as in (5b). It is therefore difficult to analytically derive insights concerning the effect of swaps and swaps portfolios upon system integrity.

However, the sensitivity of survival to small changes in physical default status can be obtained with a local linearisation of the survival switch equations, for small changes in physical default status. If survival itself is binary, one can instead consider expected survival conditional upon default $\boldsymbol{d}$, as in specification (5b). The differentials $\delta\boldsymbol{s}$ refer to small changes in expected agent survival as a consequence of small changes in physical default status $\delta\boldsymbol{d}$. Alternatively, one can operate via the fuzzy version (5c) of the binary survival function, in which case actual survival is modelled.

*4.1 Differentials*

Formal differentials of net income with respect to default status (vector $\boldsymbol{d}$) and expected survival (here, vector $\boldsymbol{s}$) can be derived using expression (3b). Denote by $\frac{\partial \boldsymbol{y}}{\partial \boldsymbol{x}'}$ the matrix whose i,j'th element is $\frac{\partial y_i}{\partial x_j}$. In general, if $H$ is a matrix of constants and $\boldsymbol{y} = H\boldsymbol{x}$ then $\frac{\partial \boldsymbol{y}}{\partial \boldsymbol{x}'} = H$ and the differentials are given by the column vector $\delta\boldsymbol{y} = H\delta\boldsymbol{x}$. Applying this to expression (3b) gives the derivatives:

(8a) $$\frac{\partial \boldsymbol{y}}{\partial \boldsymbol{s}'} = \hat{C}\hat{D}(A - \hat{A})'$$

(8b) $$\frac{\partial \boldsymbol{y}}{\partial \boldsymbol{d}'} = -\hat{B}\hat{R}\hat{C} - (A - \hat{A} + \hat{B})\hat{C} + diag(\boldsymbol{e}_j{}'(A - \hat{A})'\boldsymbol{s})\hat{C}.$$

In expression (8b) the representative vector $\boldsymbol{e}_j$ appearing in the diagonal matrix elements is the $j$'th column of the identity matrix $I$; so the $j$'th diagonal element is $\boldsymbol{e}j'(A - \hat{A})'\boldsymbol{s}$. The differential vector for net end of period income is then obtained as

(8c) $$\delta y = \frac{\partial \boldsymbol{y}}{\partial \boldsymbol{s}'}\delta\boldsymbol{s} + \frac{\partial \boldsymbol{y}}{\partial \boldsymbol{d}'}\delta\boldsymbol{d}.$$

As expected (or fuzzy formulation) survival $s_j = u_j(y_j + k_j; \sigma_j)$, it follows that

(8d) $$\delta s_j \approx \theta_j \delta y_j,$$

where $\theta_j = u'(y_j + k_j; \sigma_k)$ also depends upon current net income $y_j$ . Note with reference to figure 2a that the changes $\delta y_j$ will have to be small to avoid approximation overshooting, especially if the switch is quite sharp (small $\sigma_{j)}$ .

Combining expressions (8a-d), the differential for expected survival, conditional upon physical defaults $\boldsymbol{d}$, can be written in the form

(9) $$\delta \boldsymbol{s} = -[I - \hat{\theta}\hat{C}\hat{D}(A - \hat{A})']^{-1} \hat{\theta}[\hat{B}\hat{R} + (A - \hat{A} + \hat{B}) - diag(\boldsymbol{e}_j{}'(A - \hat{A})'\boldsymbol{s})]\hat{C}\delta\boldsymbol{d},$$

assuming provisionally that the inverse exists. The elements of the diagonal matrix $\hat{\theta}$ depend upon $\boldsymbol{y}$ and therefore on current state $\boldsymbol{s}$. The extent to which survival prospects diminish in response is given by expression (8). As net income $y_j$ approaches the default reserve point (-$k_j$), the slope $\theta_j$ changes rapidly and the changes $\delta d_j$ must be smaller to avoid overshooting in the approximation.

The intended interpretation of expression (9) is to gauge the sensitivity of counterparty income and survival to small differences in credit status of the primary physical bonds. A secondary interpretation could run in terms of the effects on survival of small real time changes in $\boldsymbol{d}$, intra-period in nature, in the sense that there is no opportunity for renegotiation among the parties. In this case equation (9) amounts to a local linearization of the differential form of (3b,5b). Sensitivity to further credit downgrades depends upon the existing extent to that point (via $\hat{D}$) and also of the existing counterparty survival or expected survival ($\boldsymbol{s}$).

*4.2 Application to the block recursive model*

The effect of the inverse appearing in expression (9) can be analysed in the context of the block recursive model (6b,7), with two types of agent, type 1 being the credit protection buyers protecting the physical debt exposures, and type 2 the credit protection sellers. In this case the required inverse is given by

(10) $$[I - \hat{\theta}\hat{C}\hat{D}(A - \hat{A})']^{-1} = \begin{bmatrix} I & (\hat{\theta}\hat{D}\hat{C})_{11} A'_{21} \\ O & I \end{bmatrix}.$$

Substituting (10) into (9), the differentials are given by

(11a) $$\delta \boldsymbol{s}_1 = \hat{\theta}_{11}\{diag(\boldsymbol{s}_2{}' A_{21}\boldsymbol{e}_{ja}) - [\hat{B}_{11}(I + \hat{R})_{11} + (\hat{D}\hat{C})_{11} A_{21}{}' \hat{\theta}_{22} A_{21}]\}\hat{C}_{11}\delta \boldsymbol{d}_1$$

(11b) $$\delta \boldsymbol{s}_2 = -\hat{\theta}_{22} A_{21} \hat{C}_{11} \delta \boldsymbol{d}_1 .$$

Combining (10a and b) gives the interdependence between survival of groups 1 and 2:

(12) $$\delta \boldsymbol{s}_1 = \hat{\theta}_{11}[diag(\boldsymbol{s}_2{}' A_{21}\boldsymbol{e}_{ja}) - \hat{B}_{11}(I + \hat{R})_{11}\hat{C}_{11}]\delta \boldsymbol{d}_1 + (\hat{\theta}\hat{D}\hat{C})_{11} A_{21}{}' \delta \boldsymbol{s}_2 .$$

The vector $\boldsymbol{e}_{ja}$ appearing in expressions (11a, 12) is the *j*'th column of the $n_1 \times n_1$ identity matrix. The diagonal element $\boldsymbol{s}_2' A_{21} \boldsymbol{e}_{ja}$ refers to the insurance payouts on the CDS in favour of a counterparty *j* that belongs to group 1, those who have bought insurance. These payouts will take place provided group 2 individuals are currently in survival state.

However, the survival state of group 2 is itself changing in response to credit events (equation (11b)), and this will adversely affect the survival prospects of group 1. Expression (12) gives the causal sequence in survival space. Any weakness in type 1 party *j*'s physical debt ($d_j > 0$) will expose that party to a risk arising from the potential survival of not only its counterparty, but to the physical debt defaults of others of group 1. The latter arises because of a risk to the survival of its counterparty in group 2, which might have insured other group I defaults.

Borrowing terminology from the econometrics of simultaneous models, the two versions (11), (12) can be respectively referred to as the 'reduced form' and 'structural' representations. The reduced form is useful to quantify aspects of contagion, in which deterioration in one debt physical can impact on the survival of apparently unconnected parties. The reduced form differential is vectored via the matrix $A_{21}' \hat{\theta}_{22} A_{21}$ appearing in the right hand side of (11a). If we assume that the intrinsic speeds of survival response to net income are locally the same for all swap writers, then $A_{21}' \hat{\theta}_{22} A_{21} \propto A_{21}' A_{21}$ . The columns of $(A_{21}' A_{21})$ capture the indirect effects of physical debt default upon risk protection buyers, arising from failure in their credit risk protection counterparties (group 2). Thus suppose there is a progressive downgrade in physical debt *j*. The *j*'th column of $(A_{21}' A_{21})$ selects for the marginal effects upon not only upon party *j* but indirectly upon all other risk protection buyers, even those not directly exposed to physical debt *j*. The off diagonal elements give the extent to which physical credit defaults diffuse through the system to adversely impact on other risk protection buyers.

*4.3 Exposure shares and concentration*

Using expression (7b) for the recursive model, the contingent exposure shares for the type 2 agents (credit protection sellers) are given by

$$\boldsymbol{w} = \frac{1}{\boldsymbol{1}' A_{21} \hat{C}_{11} \boldsymbol{d}_1} A_{21} \hat{C}_{11} \boldsymbol{d}_1,$$

where the summation vector $\boldsymbol{1}$ is here of order $n_2$, the number of credit protection sellers. Thus $w_i$ refers to the share of total system CDS exposures that will be carried by credit protection seller *i*. A highly concentrated system would be one where just one credit protection seller is carrying a high proportion of the total cost of physical debt defaults in the system.

In general, the exposure shares depend upon the state of physical debt defaults (values of $\boldsymbol{d}_1$). Credit protection sellers will be affected differently depending upon the weighting of their portfolios as between the different credit protection buyers. In one special case, however, the exposure shares $\boldsymbol{w}$ will always be constant, independent of default status. This is where the columns of $A_{12}$ are collinear, meaning that credit protection buyers all deal with the counterparties in the same proportions. Using $A'\boldsymbol{1} = \boldsymbol{1}$ for the full system, the submatrix $A_{21}$ can then be cast in the form

(13) $A_{21} = \boldsymbol{w}\boldsymbol{1}'(I - \hat{A}_{11})$.

Because $A_{12}$ has by assumption rank 1, its rows are also collinear[7], meaning that credit protection sellers portfolios are also effectively the same, apart from a scale factor.

To see the effect of this on stability, suppose that adjustment speeds are locally the same among credit protection buyers and sellers, respectively:

$$\hat{\theta}_{11} = \varphi_1 I; \hat{\theta}_{22} = \varphi_2 I .$$

Assume also that $\hat{A}_{11} = O$, meaning that group 1 credit protection buyers cover the full amount of their potential physical default costs through the use of CDS with group 2 sellers. Referring back to expression (11a), the reduced form term incorporating the counterparty effect on group 1 buyer survival, i.e. the last right hand side term, can be cast as

(14) $$-\hat{\theta}_{11}(\hat{D}\hat{C})_{11}A_{21}'\hat{\theta}_{22}A_{21}\hat{C}_{11}\delta\boldsymbol{d}_1 = -\varphi_1\varphi_2\boldsymbol{w}'\boldsymbol{w}\begin{bmatrix} d_1c_1 \\ d_2c_2 \\ \vdots \\ d_{n_1}c_{n_1} \end{bmatrix}(\textstyle\sum_{i=1}^{n_1} C_i\delta d_i) .$$

The significance of the common scalar term $(\sum_{i=1}^{n_1} C_i\delta d_i)$ is that any group 1 buyer is now equally exposed to physical default deterioration in every other group 1 buyer, simply because they all rely on the credit protection sellers in the same proportions. The effect only starts expressing itself when the group 1 buyer is already experiencing some of its own default problems, and it arises because the group 2 swap counterparties to whom it has contracted protection are themselves exposed to other group 1 deteriorations. In other words,

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[7] In this case, the seller portfolios are proportional to the diagonal elements of $I - \hat{A}_{11}$, i.e. $\{1 - \alpha_{ii}; i = 1,2,...n_1\}$.

the group 1 buyer exposure is contingent, but it fails just when it is needed. One could imagine a sequential process in real time in which an initial deterioration in physical credit #1 is followed shortly by another, so the adverse multiplier $(\sum_{i=1}^{n_1} C_i \delta d_i)$ starts to build up, serving to amplify the effects of any further physical deterioration. As counterparties begin to default on their swap obligations, there is a reassessment of the safety of the underlying physicals. Further downgrades on the physicals aggravate the adverse multiplier. Widespread physical and counterparty defaults ensue. Eventually the system as a whole collapses.

The effect is mitigated if each credit risk protection buyer can spread its counterparty exposures to many different sellers. To illustrate, suppose that there are just two credit protection sellers and each buyer deals with them in the same proportions, so that $w_1 = w_2 = \frac{1}{2}$. Then $\boldsymbol{w}'\boldsymbol{w} = \frac{1}{2}$ in equation (14). But if there are 10 credit protection sellers, with equal shares of a tenth of each buyer's business, then $\boldsymbol{w}'\boldsymbol{w} = \frac{1}{10}$, only a fifth of the implosion speed. Thus credit protection seller collinearity is not such a problem if there are many sellers relative to buyers, all sharing a relatively small proportion of the risks. Having more players in the network is conducive to stability (Allen and Gale 2000), but at issue is really a ratio, that a larger number of credit protection sellers share the burden of a smaller number of credit protection buyers.

If there have to be a smaller number of credit protection sellers, the ideal situation would be where the columns of $A_{21}$ (risk protection buyer portfolios) are orthogonal to one another, for then the off diagonal elements of the matrix are zero, and contagion is unable to spread among the credit protection buyers. This cannot be arranged if there are more credit risk sellers than buyers ($n_1 > n_2$), for in this case collinearity between columns is inevitable. A second best situation might be where $A_{21}$ is block diagonal of the general form

$$A_{21} = \begin{bmatrix} \Gamma_{11} & O \\ O & \Gamma_{22} \end{bmatrix}.$$

In this case, the second subgroup of credit protection buyers (here conformal to $\Gamma_{22}$) are insulated from physical defaults of the first subgroup.

Other things being equal, it is evidently a bad idea to have just a few major insurers, even independently of the size of their exposures. Such a system will collapse quicker when things start to go wrong with the underlying physicals. This suggests that market stability is promoted by ensuring that dominant players do not emerge among the insurers and by educating insurance buyers on the dangers of relying on the same sellers as everyone else. To put it differently, both buyer and seller portfolios need to be well diversified and ideally also a small proportion of each buyer exposure.

## V Policy implications

Preceding analysis has explicitly or implicitly identified a number of sensitive areas for the stability of a swaps-vectored debt insurance market:

(a) Whether or not buyer or seller portfolios are either well diversified or else formally decomposable, so that a failure in just one particular physical debt or debt class cannot threaten the survival of apparently unconnected parties.

(b) Whether adequate reserves ($k_j$) are set aside against both physical defaults and more particularly, against claims on credit insurers. Reserves govern the incidence and local speed ($\theta$) of survival deterioration.

(c) Whether limits on positions should be imposed, reflecting parameters such as ($\beta_j$), the extent to which naked swaps can be instituted, and perhaps also the formal size of the swaps associated with any particular physical debt issue.

(d) Whether CDS are correctly priced ($\pi_j$), in the light of both inherent probabilities of default on the underlying physicals, but also in response to the survival prospects of swap counterparties.

(e) The extent to which more relaxed attitudes towards physical debt magnitudes are induced by the apparent existence of insurance type derivatives (effectively the size $C_j$ of issues and exposures).

Points (a) - (c) are concerned with network integrity and robustness. Current practice is to require over the counter (OTC) swap counterparties to post reserves, typically with the organising bank. In the wake of the subprime crisis, it has been argued (e.g. Hull ((2010)), that all CDS, including those formally OTC in nature, should be organised via a central clearing house, which would hold margin and settle on the basis of netting. Duffie and Zhu (2009) argue for a more embracing clearing house that would cover all types of swaps, including vanilla interest rate swaps, which have intrinsically smaller potential losses than do CDS.

Regulation via clearing house arrangements does not in itself solve the issue of just how much margin should be required; nor does it convey much of a picture about the state of the network system as a whole, and its further systemic linkages. A regulator looking for diagnostics as to network safety might look at network concentration statistics, e.g. the vector $\boldsymbol{w}$ of exposure shares of section 4.3. This might be thought a demanding task, as it requires some knowledge of the reference network as a whole, which may be international in scope. On the other hand, market indices such Itraxx or Markit span many of the swaps of primary interest for system stability, so the effective input-output table *(A)* could span these as a representative sample of CDS names in the wider CDS network. Reporting requirements could see counterparties involved in swaps on these names providing the required exposure information. Regulators might further use such information to set required reserve ratios, certainly where formal clearing house margin arrangements do not exist.

With respect to point (d), pricing and volume are related via portfolio equilibrium conditions. Thus equation (7b) for the recursive model, could be written in an equivalent risk neutral world as

$$(15) \quad \rho \boldsymbol{k} = -A_{21}\hat{C}_{11}\boldsymbol{d}_1^Q + A_{21}\boldsymbol{\pi}_1 \ ,$$

where ρ is the risk free interest rate. In the equivalent risk neutral world (Q), the left hand side of (15) refers to warranted returns on required reserves of credit protection sellers. The term $\boldsymbol{d}_1^Q$ incorporates the adjusted probability of downgrade or default in the risk neutral world, and $\boldsymbol{\pi}$ contains the swap premiums, expressed here in gross terms (as distinct from the spread over the safe asset, the usual convention). Equation (15) is a portfolio equilibrium condition that relates the supply and demand of CDS (incorporated as $A_{21}$), to their price and the reserves devoted to back them. For a given swap premium, a lower level of reserves should inflate the risk neutral default probability and hence diminish the volume of swaps written.

In conventional market practice swap premiums have been determined by comparison with actual or potential asset swap arbitrages involving long or short portions in the underlying physical, augmented with interest rate swaps to shift between fixed and floating funding costs (e.g. Choudry (2004a,b, 2006a.b), Das and Hanouna (2006)). Liquidity constraints in the market may indicate some departures from time to time.

However, the basis spread between the physical debt and the CDS should also involve assessment and pricing of counterparty risk. In practice, this function has been implicitly devolved to the major credit rating agencies. Thus an AAA or Aaa rating has been treated as a sufficient statistic for a very low level of counterparty risk, and hence a reliable counterparty. In turn, this should impose a responsibility on the part of the credit rating agency as implicit certifier to examine the implicit as well as explicit linkages that might impact on the survival of any given counterparty. In its full generality, this would encompass the global linkages of all other counterparties, and the default risks they have underwritten – in effect, the full system matrix $A$, the size of the risks ( $B$, $C$ elements), and the reserves ($\boldsymbol{k}$) that underpin them. That is likely to be infeasible, so a second best recourse might be to ensure that any given counterparty, whether buyer or seller, has an adequately diversified portfolio, with limits on exposure shares.

In carrying out such functions, private credit rating agencies overlap with regulatory responsibility, just as their judgments were viewed as sufficient for bank capital adequacy in the Basel Standard model; an implied agency relationship. However, determination of reserve adequacy is a central responsibility of financial regulatory agencies, and this should apply to any swap counterparty, particularly where a bank is a counterparty, or is indirectly so by virtue of network paths. There is a margin of choice as to whether CDS reserves are to be set and administered by a centralised clearing house system for swaps, or simply by regulatory edict, but the market in any case needs to know where it stands in making decisions as to which counterparties to deal with or what price to ask. It is not clear whether clearing house rules and margin requirements would cover portfolio diversification or the lack of it. The best way to do this is for the financial regulator to require disclosure of counterparties, and metrics for portfolio diversification, with market exposure shares to be within required limits. Concentration monitoring should apply to any counterparty, whether generating risk or at risk, and whether banks, investment funds, insurance companies or hedge funds. In the meantime, the structural predisposition to contagion and collapse remains.

## References


Allen, F. and D. Gale (2000) Financial contagion, *Journal of Political Economy*, 108(1), 1-33.

Choudhry, M. (2004a) The credit default swap basis: analysing the relationship between cash and synthetic credit markets, *Journal of Derivatives Use, Trading and Regulation*, 10, 8-26.

Choudhry, M. (2004b) *Fixed Income Markets: Instruments, Applications, Mathematics*, Wiley.

Choudhry, M. (2006a) Revisiting the credit default swap basis: further analysis of the cash and synthetic credit market differential, *Journal of Structured Finance*, 11,4, 21.

Choudhry, M. (2006b) *The Credit Default Swap Basis*, Bloomberg Press.

Das, S.R. and P. Hanouna (2006) Credit default swap spreads, working paper, Villanova University.

Duffie, D., and H. Zhu (2009) Does a central clearing counterparty reduce counterparty risk? Working paper 2022, Stanford University School of Business.

Furfine, C. (2003) Interbank exposures: Quantifying the risk of contagion, *Journal of Money, Credit, and Banking*, 35(1), 111–128.

Gai, P. and S. Kapadia (2007) Contagion in Financial Networks, working paper, Bank of England.

Hull, J. (2010) OTC derivatives and central clearing: Can all transactions be cleared? Working paper, Trotman School of Management, University of Toronto

Ishikawa, T. (2009) *How I caused the credit crunch*, London: Icon Books.

Lewis, M. (2010) *The big short: inside the doomsday machine*, Penguin/Allen Lane.

Loeffler, G. and Posch, P.N. (2007) *Credit risk modelling using Excel and VBA*, Chichester UK: Wiley.

Longstaff, F., S. Mithal, and E. Neis (2005) Corporate yield spreads: default risk or liquidity? New evidence from the credit default swap market, *Journal of Finance* 60, 5, 2213-53.

Posch, P.N. (2007) *On the dynamics of credit risk: an econometric analysis*, PhD Thesis, Faculty of Mathematics and Economics, Ulm University, www.dissertation,de

Rajan, A., G.McDermott and R. Roy (2007) *Structured credit handbook*, Holboken N.J.: J. Wiley.

Zadeh, L. (1965) Fuzzy sets, *Information and Control*, 8(3):338-353.

Zadeh, L.A. (1996) *Fuzzy sets, fuzzy logic and fuzzy systems: selected papers*, River Edge, NJ: World Scientific Publishing.